$GL(n,\mathbb{Z}) :=$ group of $n\times n$ invertible integer matrices A

$\underline{n=2}$

$a := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $b := \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

Claim: a & b generate a free subgroup of $GL(2,\mathbb{Z})$!

Idea: Let's study the action of a & b on $\mathbb{Z}^2$.

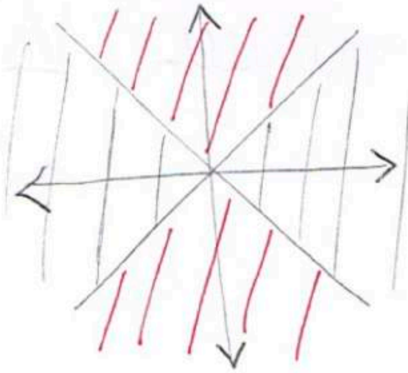
$GL(2,\mathbb{Z}) \circlearrowright \mathbb{Z}^2$ by matrix-vector multiplication.

eg. $a\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. (for all $k \in \mathbb{Z}$)

#1. $a^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \ldots, a^k = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix}$.

#2. when $|x|<|y|$, $a^k\cdot\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2ky \\ y \end{pmatrix}$

and $|x+2ky| \geq |2ky - x| > |y|$.



※ Things in the red zone get sent into things in the black zone, and they stay there too!

#3. b does the opposite, sending things in the black zone to things in the red zone.

#4. How does this help us? PING-PONG!

Recall the definition of a group action.

※ $I_2 \cdot v = v$ for all $v \in \mathbb{Z}^2$.

$\therefore$ If some word in a & b is equal to I_2, then that word fixes everything!

LC

Is a^3b^2a a relator?

→ Choose any v in the black zone and see that $(a^3b^2a)\cdot v$ is in the red zone!

That is, $(a^3b^2a)\cdot v \neq v$,

so $a^3b^2a \neq I_2$, i.e., a^3b^2a is not a relator!

• In fact, $\langle a,b\rangle$ has no relators!

Lemma ("Ping-Pong for two players")

• G generated by a & b.

$a_1, a_2, \ldots, a_n$ g

• $G \curvearrowright X$ nonempty.

$a_i^k(X_j) \subseteq X_i$

• Have $X_a, X_b \subseteq X$: $a^k(X_b) \subseteq X_a$ & $b^k(X_a) \subseteq X_b$

$1 \quad 2 \quad n$

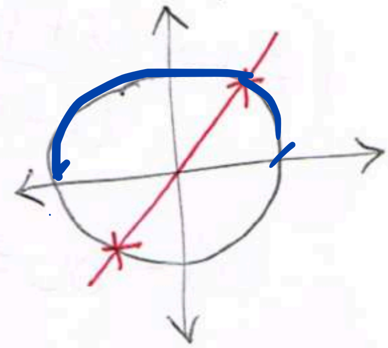
$\Rightarrow \cancel{G \cong} F_2$ ← free group of rank 2.

$i=1$ B_1 B_g

1D

$\mathbb{RP}$:= space of 1-dimensional subspaces of $\mathbb{R}^2$ (as a vector space)

$\cong$ unit circle in $\mathbb{R}^2$ with antipodal points identified.



$\mathbb{CP}$:= space of 1-dimensional subspaces of $\mathbb{C}^2$.

$\cong$ "Riemann sphere"

↑ This is a complex manifold!

Möbius Transformations

$$z \mapsto \frac{az+b}{cz+d} \quad \text{where } ad-bc \neq 0.$$

$$\left(-\frac{d}{c} \mapsto \infty, \quad \infty \mapsto \frac{a}{c}\right)$$

$$* \left(z \mapsto \frac{a'z+b'}{c'z+d'}\right) \circ \left(z \mapsto \frac{az+b}{cz+d}\right) = \left(z \mapsto \frac{(aa'+b'c)z+(a'b+b\ldots}{(c'a+d'c)z+(bc'+d\ldots}\right.$$

E

Have a natural identification of Möbius transformations with 2x2 matrices over $\mathbb{C}$!

$$\left(z \mapsto \frac{az+b}{cz+d}\right) \longleftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$$ad - bc \neq 0 \iff \in GL(2,\mathbb{C}).$$

almost...

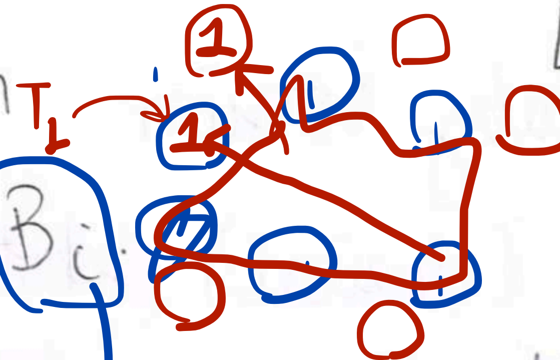
not well-defined because $\dfrac{az+b}{cz+d} = \dfrac{(\boxed{2}a)z+(\boxed{2}b)}{(2c)z+(\boxed{2}d)}$

Solution: $\boxed{GL(2,\mathbb{C}) / \mathbb{C}^{\times} I_2} \cong \boxed{PGL(2,\mathbb{C})} \cong \boxed{PSL(2,\mathbb{C})}.$

Schottky groups

· 2g disjoint open disks ~~circles (with disjoint interiors)~~ $A_1, B_1, \ldots, A_g, B_g$.

T_i Möbius transformation so that $T_i(\boxed{A_i}^c) \subseteq B_i$.



- Since the set of all Möbius transformations is a group, the T_i generate a subgroup of this group, and under the identification of this group with $PSL(2,\mathbb{C})$, a subgroup of $PSL(2,\mathbb{C})$!

Such a subgroup is called a "classical Schottky group"

Theorem (Maskit 1967)

theorem [1], and the planarity theorem [3]. *A finitely generated Kleinian group G is a Schottky group if and only if G is free, and every element of G other than the identity is loxodromic* (hyperbolic transformations are included among the loxodromic).

LG

Baby version:

(classical) Schottky groups are free!

Proof. Apply the Ping-Pong lemma. $\square$